

Computer Modeling and Analysis of Plane Wave Propagation Through Plasmas

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Abstract

A technique for modeling plane wave propagation through inhomogeneous media and plasma layers with different electron densities is presented. The technique is based on transmission line theory. Therefore, a general technique is introduced for solving complex transmission line systems. A data structure and algorithm for representing, and *simultaneously* solving for all nodes within the transmission line network is presented. The method is based on representing the network as a recursive tree structure and solving for the voltage, current, and impedance at each node using recursive programming techniques. First, all frequency dependent parameters within the tree structure are updated, then in a post-order traversing of the tree, the impedances at each node are computed followed by a pre-order traversing of the tree to compute node voltages and currents. For plane wave propagation, the reflection coefficient, the electric field and magnetic field are computed. The method is applied to normal incidence, but can easily be extended to oblique incidence.

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I. Introduction

In this paper we present a program for computer modeling and simulation of plane wave propagation in inhomogeneous media. The technique is based on drawing an analogy between plane wave propagation and transmission line theory. Next an algorithm for solving complex transmission line networks is presented. This of course leads to the algorithm for solving for plane wave propagation. The network is represented in the computer by a recursive binary tree data structure. Using recursive programming techniques, the node voltage current, and impedance at each node within the tree structure is computed. For plane wave propagation, the reflection coefficient, the electric field and magnetic field are computed. In this manner the frequency response of the network, from the source node to the receiving node is computed. The impulse response or the pulse response of the network is then calculated from the frequency response using Fast Fourier Transforms.

Computer programs for modeling transmission line networks have been written using ABCD parameters [1]. In this paper a new technique in which the frequency response is simultaneously obtained at all nodes within the network is presented.

II. Plane Wave / Transmission Line Analogy

Consider the plane wave propagation problem illustrated in Fig. 1. For this geometry,

$$\begin{aligned} E_y = E_z = 0 & & H_x = H_z = 0 \\ \frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0 & & \frac{\partial H_y}{\partial x} = \frac{\partial H_y}{\partial y} = 0 \end{aligned} \quad (1)$$

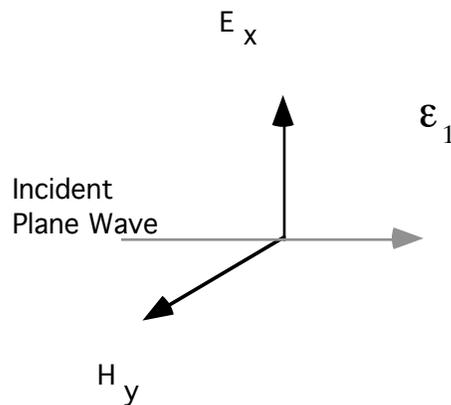


Figure 1. Plane Wave Propagation

Maxwell's equations reduce to

$$\frac{dE_x}{dz} = -j\omega\mu H_y \quad \frac{dH_y}{dz} = -j\omega\epsilon E_x \quad (2)$$

These equations are similar to the differential equations for lossless transmission lines,

$$\frac{dV}{dz} = -j\omega LI \quad \frac{dI}{dz} = -j\omega CV \quad (3)$$

Hence, the theory used to describe propagation in transmission lines applies equally well to plane-wave propagation. Next, consider the discontinuity at the boundary between two different medium with different dielectric constants. See Fig. 2. This condition is similar to the discontinuity when two different transmission lines with different characteristic impedances are connected to each other as shown in Fig. 3.

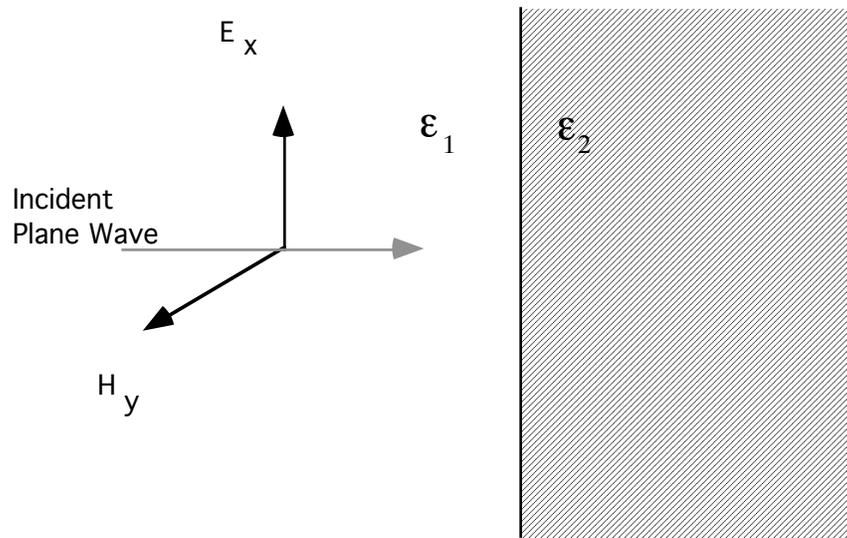


Figure 2. Plane wave discontinuity

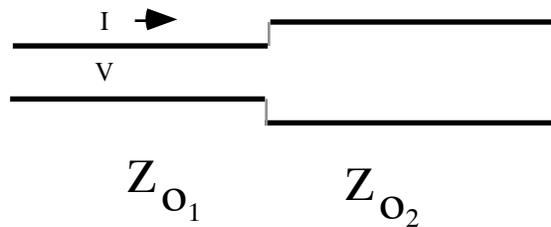


Figure 3. Transmission line discontinuity

The analogy between transmission lines and plane wave propagation (normal incidence) is summarized in Table 1. Note that the mismatch at the discontinuity is represented by a reflection coefficient for each case.

Table 1. Plane Wave/ Transmission Line Analogy

<i>Transmission Line Quantity</i>	<i>Symbol or Eq</i>	<i>Plane-Wave Quantity</i>	<i>Symbol or Eq</i>
Voltage	V	Electric field E_x	E_x
Current	I	Magnetic field H_y	H_y
Inductance per unit length	L	Permeability μ	μ
Capacitance per unit length	C	Permittivity ϵ	ϵ
Characteristic impedance	$Z_0 = \sqrt{\frac{L}{C}}$	Intrinsic impedance $\eta = \sqrt{\frac{\mu}{\epsilon}}$	$\eta = \sqrt{\frac{\mu}{\epsilon}}$
Phase-shift coefficient	$\beta = \omega\sqrt{LC}$	Phase-shift coefficient $\beta = \omega\sqrt{\mu\epsilon}$	$\beta = \omega\sqrt{\mu\epsilon}$
Velocity of propagation	$v = \frac{1}{\sqrt{LC}}$	Velocity of propagation $v = \frac{1}{\sqrt{\mu\epsilon}}$	$v = \frac{1}{\sqrt{\mu\epsilon}}$
Reflection Coefficient	$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$	Reflection coefficient between e_1 and e_2 $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$
Incident wave power	$P^+ = \frac{ V^+ ^2}{2Z_0}$	Incident wave power $P^+ = \frac{ E_x^+ ^2}{2\eta}$	$P^+ = \frac{ E_x^+ ^2}{2\eta}$

Reference: Robert G. Brown, Robert A. Sharpe, William L. Hughes, Robert E. Post, *Lines, Waves, and Antennas*, Second Edition, John Wiley and Sons, New York, 1973

Table II. Summary of Plasma Parameters of Interest

Permittivity, n electrons/cm ³	$\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 \left(1 - \frac{81 n}{f_{\text{kHz}}^2} \right)$
Intrinsic Impedance	$\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{\eta_0}{\sqrt{1 - \frac{81 n}{f_{\text{kHz}}^2}}}$
Propagation Constant	$\gamma = \alpha + j \beta = j \omega \sqrt{\mu_0 \epsilon_0} \sqrt{1 - \frac{81 n}{f_{\text{kHz}}^2}}$
Group Velocity	$v_g = \frac{d\omega}{d\beta} = c \sqrt{1 - \frac{81 n}{f_{\text{kHz}}^2}}$
Critical Frequency, e electron charge, m electron mass	$\omega_p^2 = \frac{e^2 n}{m \epsilon_0}$
Introduction of loss due to collision Frequency ν .	$\epsilon = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2 (1 - j \frac{\nu}{\omega})} \right]$
Attenuation Constant	$\alpha = \frac{\eta_0 n e^2 \nu}{2m (\omega^2 + \nu^2)}$

Based on the above discussion we can solve the problem of plane wave propagation through the system shown in Fig.4, by solving the equivalent transmission line problem shown also. Therefore, we will focus on solving general transmission line networks and return to plane wave propagation later in the paper.

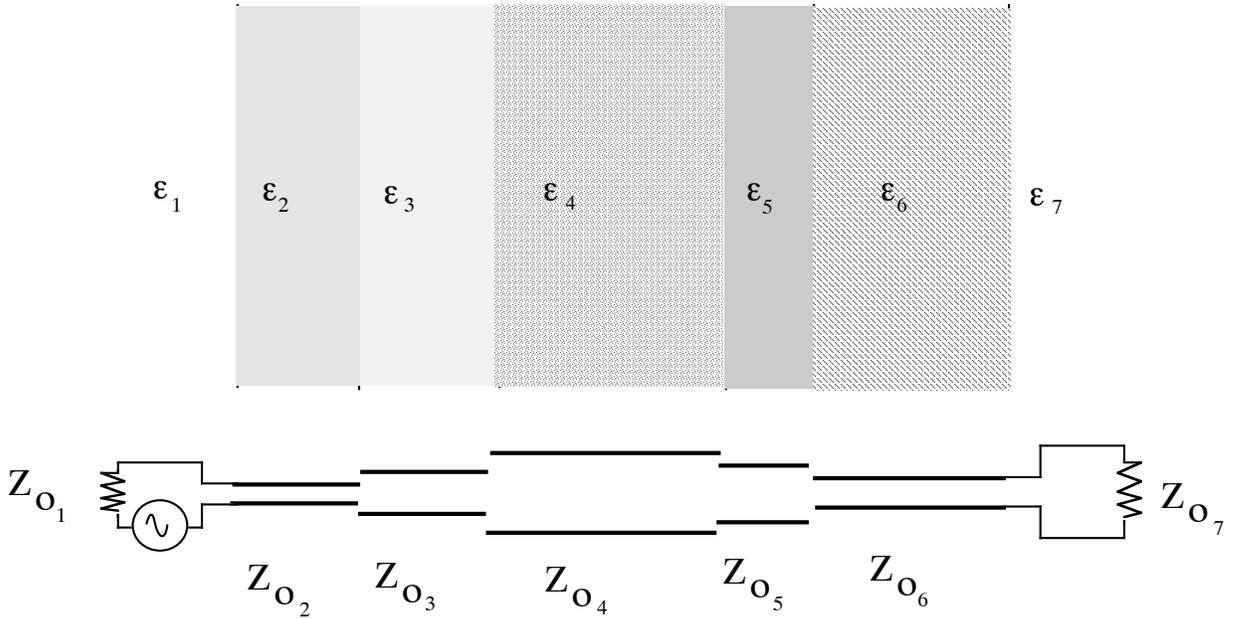


Figure 4. Modeling propagation through multiple layers as transmission line sections.

III. Transmission Line Networks

Consider the basic problem of simulating pulse transmission through a loaded transmission line. Assume that the pulse of interests is bandlimited with a cutoff frequency of f_c . We can obtain the pulse response by first computing frequency response of the network at equal intervals, then we perform a complex multiplication of the frequency response of the pulse and the transmission line network as calculated, and finally the inverse FFT of the result yields the time domain pulse response. Actually, the impulse response can also be obtained by computing the inverse FFT of the frequency response. Therefore, as a first step in calculating the frequency response of the network, we analyze the network response to a single sinusoid of frequency f_0 . Consider the loaded transmission line connected to the generator E_g through a source impedance Z_s as shown in Fig. 5 [2].

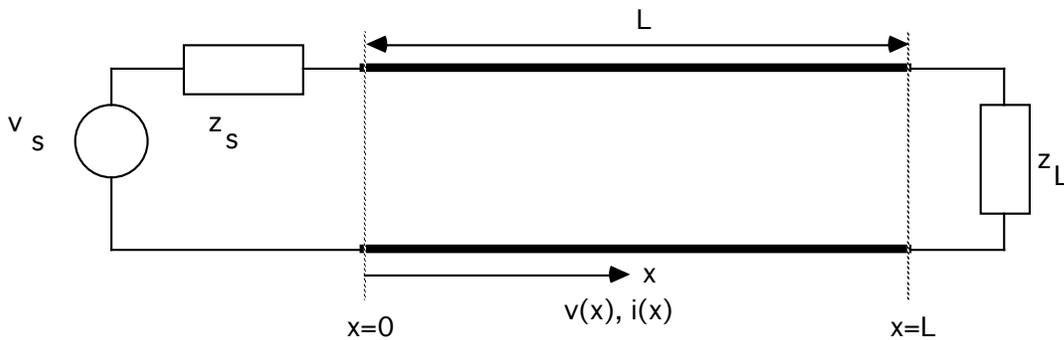


Figure 5. Generator connected to loaded transmission line

The voltage and current at any point on the transmission line can be obtained from the following expressions:

$$v(x) = \frac{v_s z_0}{z_0 + z_s} e^{-\gamma x} \frac{1 + \Gamma_L e^{-2\gamma(L-x)}}{1 - \Gamma_s \Gamma_L e^{-2\gamma L}} \quad (4)$$

$$i(x) = \frac{v_s}{z_0 + z_s} e^{-\gamma x} \frac{1 - \Gamma_L e^{-2\gamma(L-x)}}{1 - \Gamma_s \Gamma_L e^{-2\gamma L}} \quad (5)$$

In the above expressions

$$\gamma = \sqrt{(r + j\omega l)(g + j\omega c)} \quad (6)$$

is the propagation constant and

$$z_0 = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \quad (7)$$

is the characteristic impedance of the transmission line. The expressions for the source and load reflection coefficients are,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad (8)$$

$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} \quad (9)$$

The expression for $v(x)$ includes the superposition of all waves reflecting from the source and load mismatches. This can be seen by a Taylor series expansion of (4)

$$v(x) = \frac{v_s z_0}{z_0 + z_s} [e^{-\gamma x} + \Gamma_L e^{-\gamma(L-x)} + \Gamma_L \Gamma_s e^{-\gamma(2L+x)} + \Gamma_L^2 \Gamma_s e^{-\gamma(3L-x)} + \Gamma_L^2 \Gamma_s^2 e^{-\gamma(3L+x)} + \dots] \quad (10)$$

To obtain the shape of the pulse at the load we evaluate $v(L)$ at frequencies from $f=0$ to $f=fc$ in discrete steps where f_c is the cutoff frequency of the bandlimited pulse. The number of points must be a power of 2 such that the inverse FFT may be used to obtain the sampled pulse response at the load.

Consider now the case where the boundary voltage and current are known on a section of transmission line. See Fig. 2. Evaluate $v(0)$ in (4) and then compute.

$$\frac{v(x)}{v(0)} = e^{-\gamma x} \frac{1 + \Gamma_L e^{-2\gamma(L-x)}}{1 + \Gamma_L e^{-2\gamma L}} \quad (8)$$

Also

$$\frac{i(x)}{i(0)} = e^{-\gamma x} \frac{1 - \Gamma_L e^{-2\gamma(L-x)}}{1 - \Gamma_L e^{-2\gamma L}} \quad (9)$$

Thus using (8) and (9) the voltage and current can be evaluated at any point on the transmission line given the boundary voltage and current.

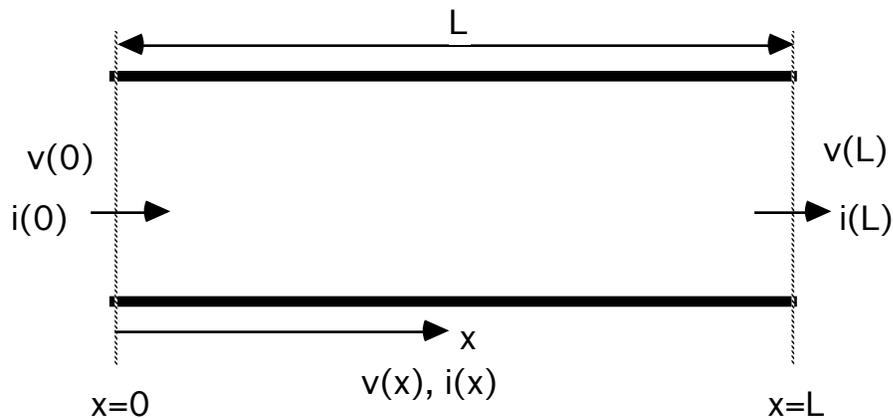


Figure 6. Section of transmission line with boundary voltages and currents

With the above preliminaries, we will examine the simple network in Fig. 7 and present a methodology for its solution. In Fig. 7, the nodes have been labeled n_1 through n_5 . To solve this network, that is to obtain the voltage and current at each node and at any location within the network, consider equation (4). This equation suggests that if the impedance at node n_1 was known then the voltage and current at node n_1 can be calculated from the generator and source impedance. Thus the first step is to obtain the impedance at n_1 . This impedance is seen to consist of the parallel combination of the impedance looking into n_5 and n_2 from n_1 .

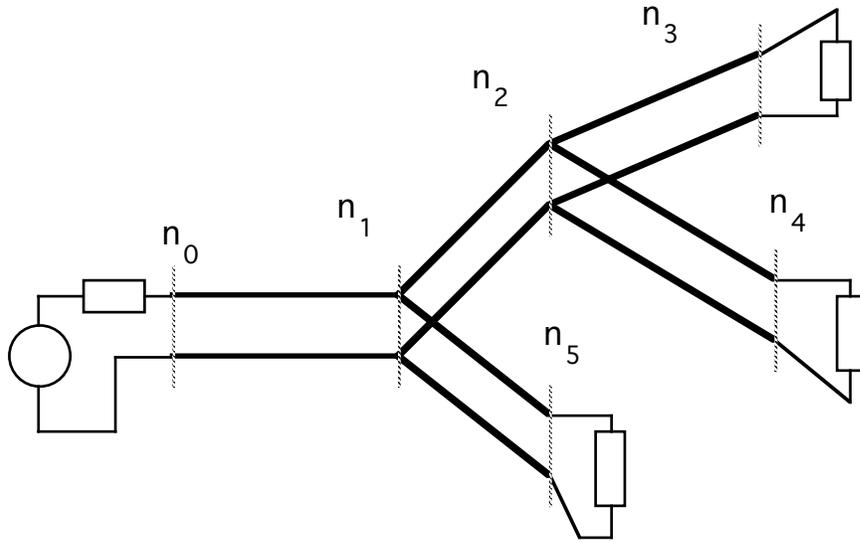


Figure 7. Example transmission line network

These impedances can be obtained by noting that (Fig. 8),

$$Z_{in}(x) = \frac{1 + \Gamma_L e^{-2\gamma(L-x)}}{1 - \Gamma_L e^{-2\gamma(L-x)}} Z_0 \quad (10)$$

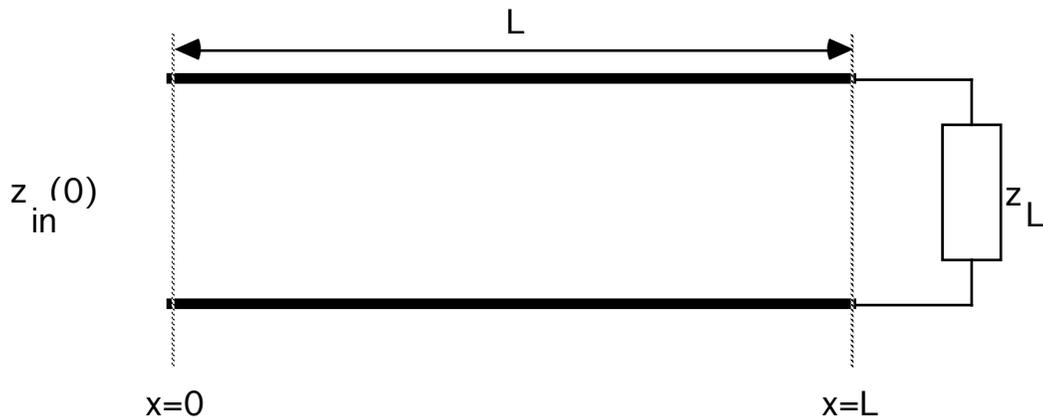


Figure 8. Input impedance of a loaded transmission line.

Thus, the first step is to calculate the impedances looking into n_3 and n_4 from n_2 . The parallel combination forms the impedance at n_2 . The impedance at n_1 is thus calculated by the parallel combination of the impedances looking into n_2 and n_5 .

Therefore, the following methodology is suggested for solving the network. In the first pass, starting from the three loaded end nodes, the impedances are calculated and the parallel combination of these impedances at the parent node forms the parent node impedance. Working

backward in this manner, the impedance at the root node (n_1 in the example) is calculated. Using (1) the voltage and current at the root node n_1 is calculated. Using (8) and (9) and the boundary voltages and currents, calculated at the parent node, the voltage and current at each node in the network can be calculated. Note that the current at each node is split into two currents flowing into each node.

In the case of propagation through layers of different medium, there are no branches. This situation is highlighted in Figure 9. In this case the impedance at node n_5 is computed first. Next, the impedance at node n_4 is computed and so forth until the total impedance looking into the network is obtained. In the next phase, the voltage and currents are computed starting from the source, node n_1 and moving towards the load.

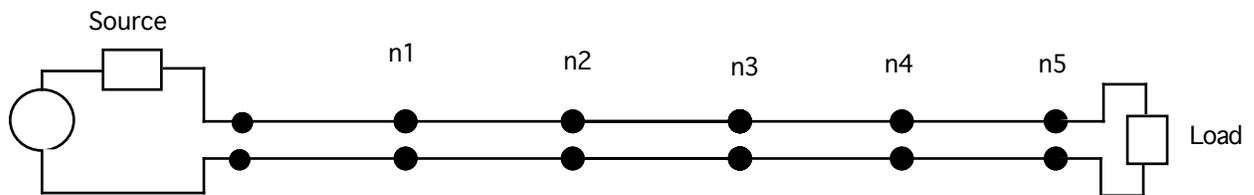


Figure 9. Network of cascaded sections.

IV. Recursive Programming and Data Structures

To introduce the algorithm for solving a complex transmission line network, we first consider the case where the network is limited to the binary tree structure shown in Fig. 10. In the figure, the generator is connected to the root of the tree through a source impedance Z_s . The tree consists of nodes which are either parents or leaves. A leaf is a node which is terminated on a load. For example, $n_3, n_4, n_6, n_7, n_9, n_{11}$, and n_{12} . Parent nodes have two branches. A left branch and a right branch. Nodes n_1, n_2, n_5, n_8 , and n_{10} are parent nodes.

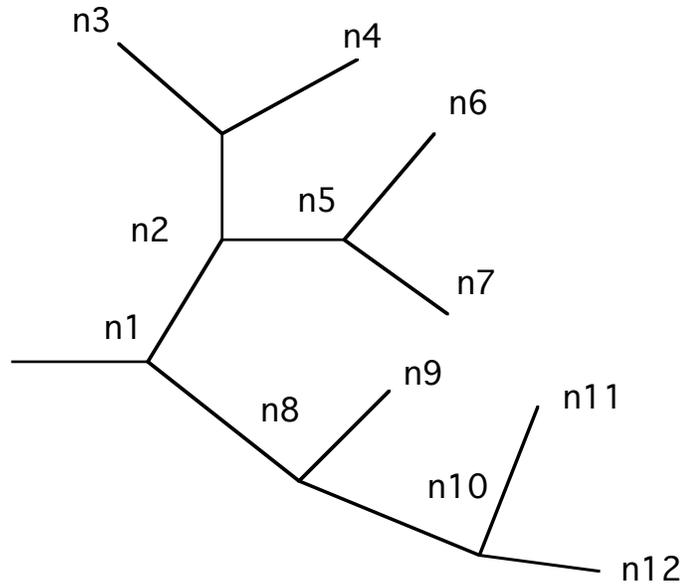


Figure 10. Transmission line network as a tree structure

In general each branch represents a transmission line with different characteristics and lengths. Each section of transmission line is associated with the node on which it terminates. Thus the section of transmission line from the generator to the root node n_1 is described in the data structure pointed to by n_1 . This concept is described below.

Each node has an associated data structure which occupies memory locations. A pointer can be defined which points to the data structure in memory. As nodes are added to the tree, memory is dynamically allocated for the data structure and a pointer is defined.

A detailed description of the algorithm to solve the binary tree representation of transmission line network is available at:

http://trantopcalc.sourceforge.net/trantopcalc_algorithm.htm

VIII. Computer Simulation Results

1. Plane Wave Propagation Through Plasma

Figure 15 shows a plasma sandwiched between two media with dielectric constants $\epsilon_r = 1.0$. A metal plate is placed against the second slab. The electron density of the plasma is $3.16 \times 10^{14} \text{ e}^-/\text{cm}^3$ corresponding to a critical frequency of 160 GHz.

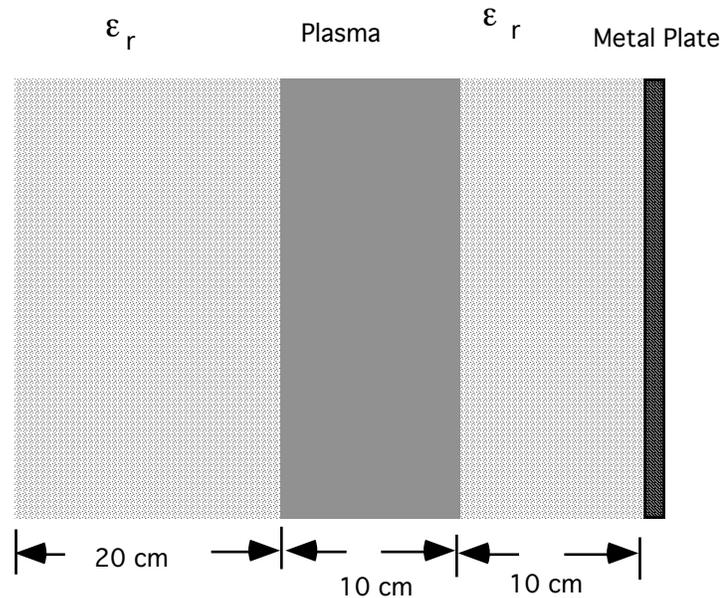


Figure 15. Plasma Propagation Model

Two cases are presented. In the first case, the impulse response over a bandwidth of $20 \times 512 = 10240$ MHz and starting frequency of 150 GHz is considered. For the second case, the same bandwidth was used but the beginning frequency was at 200 GHz. Since in both cases, 512 frequency samples were obtained, the resulting impulse response has 1024 points. Assuming free space propagation, the resolution corresponds to $2 \cdot c \cdot dt$ where $dt = 1/(1024 \cdot 20 \cdot 10^6)$ and c is the speed of light.

Since the program stores the data structure of the network, it is possible to build an array which corresponds to the distance away from the source at which various reflections occur. Thus, by taking into account the group velocity in each medium and noting that we know the length of

each layer, we can calculate the time spent in each layer. Since dt is invariant, we can calculate the distance corresponding to each sample through the following procedure:

```

/*
 * Code segment for calculating the distance corresponding to each reflected
 * sample in a layer
 */
if (!plasma) velocity =SPEED_OF_LIGHT/sqrt(epsr);
else
    velocity =SPEED_OF_LIGHT*sqrt(epsr);
    /*
    * Calculate time spent in layer
    */
time = current_P->length/velocity;
/*
 * Calculate the number of samples that are reflected from this layer
 * Factor of 2 corresponds to round trip time
 */
j = (int) (2.0 *time/dt + 0.5);
dist=0.0;
i=0;
for (jj = 1; jj <= j; jj++) {
    printf("%d %d length = %f vel = %e time = %e\n",plasmaFlag,j,
           current_P->length,velocity,time );
    dist += current_P->length/(float)j * 100.0; /* dist in cm */
    distance_A[i] = dist;
    fprintf(profile_F,"%d\t%e\t%f \n",i,dist,epsr);

    i++;
    if ( i >= npts ) break;
}

```

The impulse response corresponding to the first case has a reflection due to the plasma and the frequency span below the critical frequency. In the second case the impulse response has no reflection due to the plasma but is reflected by the metal plate as the frequency span is higher than the critical frequency.

IV. Conclusions

In this technical report, a computer program is described which simultaneously solves for all nodes within complex networks of transmission lines. A tree data structure was introduced for representing the network in the computer. Recursive procedures were presented for traversing the tree. The algorithm was used to solve for plane wave propagation over a band of a dielectric, plasma and metal plate system.

X References.

- [1] D.G. Messerschmitt, "Transmission line modeling program written in C," *IEEE Journal on Selected Areas in Communications*, Vol. SAC-2, pp. 148-153, January 1984.
- [2] Dworsky, Lawrence N, *Modern Transmission Line Theory and Applications*, John Wiley and Sons, Inc., 1979.
- [3] Gerald E. Sobelman, David E. Krekelberg, *Advanced C Techniques and Applications*, Que Corporation, Indianapolis, Indiana,, 1985.
- [4] Robert G. Brown, Robert A. Sharpe, William L. Hughes, Robert E. Post, *Lines, Waves, and Antennas*, Second Edition, John Wiley and Sons, New York, 1973

Appendix C Code for Distance Calculation

```
/*
 * This routine is a special routine that creates
 * an array of distance corresponding to lengths of
 * different segments. The calculation adjusts the length for
 * the speed of light in the medium.
 * This routine ignores the right nodes and goes all the way to the
 * load. Also a file is created for the dielectric constant as a
 * function of distance
 * at a fixed frequency.
 */
void BuildDistance(root_P, distance_A, dt, freq, npts)
struct node      *root_P;          /* pointer to node */
float            distance_A[];
float            dt;
float            freq;
int              npts;
{
float            n, epsr, eps, nu;
float            dist;
float            time, accTime, totalTime;
float            velocity;
int              plasmaFlag=0;
int              i, j, jj;
FILE            *prof_F;
struct node      *current_P;      /* pointer to node */

prof_F = fopen("profile.dat", "w");

if ( root_P == (struct node *) NULL ) RET;

totalTime = 0;
current_P = root_P;
dist=0;
i=0;
fprintf(stderr, "dt freq npts  %e %e %d\n", dt, freq, npts);
/*
 * go through layers
 */
do {
/*
 * check if plasma
 */
if (strncmp(current_P->typeName, "plasma", 6) ==0 ) {
        plasmaFlag = 1;
        nu=current_P->r;
        n=current_P->l;
        epsr = 1.0-81e6*n/(freq*freq);
        if (epsr > 0) {
                eps = EPS0* epsr;
        } else {

                eps = -EPS0* epsr;
                epsr=1.0;

        }

    }
else {

        eps=current_P->c;
```

```

        epsr = eps/EPS0;
    }
    printf("%s epsr = %e \n",current_P->name,epsr);

    if (!plasmaFlag) velocity =SPEED_OF_LIGHT/sqrt(epsr);
    else
        velocity =SPEED_OF_LIGHT*sqrt(epsr);
    time = current_P->length/velocity;
    j = (int) (2.0 *time/dt + 0.5);
    for (jj = 1; jj <= j; jj++) {
    printf("%d %d length =%f vel =%e time =%e\n",
        plasmaFlag,j,current_P->length,velocity,time);
        dist += current_P->length/(float)j * 100.0;
        distance_A[i] = dist;
        fprintf( prof_F, "%d\t%e\t%f \n",i,dist,epsr);

        i++;
        if ( i >= npts ) break;
    }

} while ( (current_P = current_P->left_P)  && ( i < npts) );
if(i<npts) {
    for(j=i; j<npts; j++) {
        dist += dt * SPEED_OF_LIGHT * 100.0;
        distance_A[j] = dist;
        fprintf( prof_F, "%d\t%e\t%f \n",j,dist,epsr);
    }
}
fclose( prof_F );
}

```

